M250 Practice Test Answer Key
chap 3.1 – 3.6

1. Find all open intervals on which the function $f(x) = \dfrac{x}{x^2+x-2}$ is decreasing.

Find open intervals on which the derivative is negative.

$$f'(x) = \frac{(x)'(x^2+x-2) - x(x^2+x-2)'}{(x^2+x-2)^2}$$

$$= \frac{x^2+x-2-x(2x+1)}{(x-1)^2(x+2)^2}$$

$$= \frac{-x^2-2}{(x-1)^2(x+2)^2} = -\frac{x^2+2}{(x-1)^2(x+2)^2}$$

Answer: $(-\infty,-2)$, $(-2,1)$ and $(1,\infty)$

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2. Find all critical numbers for the function

$$f(x) = (9-x^2)^{3/5}$$

$$f'(x) = \frac{3}{5}(9-x^2)^{-\frac{2}{5}} \cdot (-2x) = \frac{-6x}{\sqrt[5]{(9-x^2)^2}}$$

$f'(0)=0$, $f'(3)$ does not exist, $f'(-3)$ does not exist

Answer: $\{-3, 0, 3\}$

3. Find the values of x that give relative extrema for the function $f(x) = (x+1)^2(x-2)$

$$f'(x) = 2(x+1)(x-2) + (x+1)^2$$
$$= (x+1)(2x-4+x+1)$$
$$= (x+1)(3x-3)$$
$$= 3(x+1)(x-1)$$

critical numbers are -1 and 1

$$f''(x) = 3(x-1) + 3(x+1)$$
$$= 6x$$

$f''(-1) = -6 < 0$ cupped downward

Answer: $\rightarrow$ relative max. at $x = -1$

$f''(1) = 6 > 0$ cupped upward

$\rightarrow$ relative min. at $x = -1$

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4. Find all intervals on which the graph of the function is concave upward

$$f(x) = \frac{x-1}{x+3}$$

ie. where is $f'' > 0$?

$$f'(x) = \frac{(x-1)'(x+3) - (x-1)(x+3)'}{(x+3)^2}$$

$$= \frac{x+3-x+1}{(x+3)^2} = \frac{4}{(x+3)^2}$$

4. (continued)

$f'(x) = 4(x+3)^{-2}$

$f''(x) = -8(x+3)^{-3} = \dfrac{-8}{(x+3)^3}$

so $f''(x) > 0$ whenever $(x+3)^3 < 0$

i.e. $x+3 < 0 \Rightarrow$ on $(-\infty, -3)$

Answer:
$f(x)$ is concave upward on $(-\infty, -3)$

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5. Let $f''(x) = 3x^2 - 4$ and let $f(x)$ have critical numbers -2, 0, and 2. Use the Second Derivative Test to determine which critical numbers, if any, give a relative maximum.

$f''(-2) = 3(-2)^2 - 4 = 8 > 0 \Rightarrow f(-2)$ is a rel. min.
$f''(0) = 3(0)^2 - 4 = -4 < 0 \Rightarrow f(0)$ is a rel. max.
$f''(2) = 3(2)^2 - 4 = 8 > 0 \Rightarrow f(2)$ is a rel. min.

Answer: $x = 2$ gives a relative maximum.

6. Find $\lim_{x \to \infty} \frac{\sqrt{4x^2-1}}{x^2}$

$$\frac{\sqrt{4x^2-1}}{x^2} = \frac{\sqrt{4x^2-1}}{\sqrt{x^4}} = \sqrt{\frac{4x^2-1}{x^4}} = \sqrt{\frac{4}{x^2} - \frac{1}{x^4}}$$

$$\lim_{x \to \infty} \sqrt{\frac{4}{x^2} - \frac{1}{x^4}} = \sqrt{0-0} = 0$$

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7. Which of the following functions has a horizontal asymptote at $y = -\frac{1}{2}$?

a) $$\lim_{x \to \pm\infty} \frac{x^3}{1-2x^3} = \lim_{x \to \pm\infty} \frac{x^3/x^3}{\frac{1-2x^3}{x^3}} = \lim_{x \to \pm\infty} \frac{1}{\frac{1}{x^3}-2} = -\frac{1}{2}$$

So $\frac{x^3}{1-2x^3}$ has a horizontal asymptote at $y=-\frac{1}{2}$

b) $$\lim_{x \to \infty} \frac{x}{\sqrt{2x+1}} = \lim_{x \to \infty} \frac{\frac{x}{\sqrt{x}}}{\frac{\sqrt{2x+1}}{\sqrt{x}}} = \lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{2+\frac{1}{x}}} = \infty$$

So $\frac{x}{\sqrt{2x+1}}$ has no horizontal asymptotes

c) $$\lim_{x \to \pm\infty} \frac{2x^2-6x+1}{1+x^2} = \lim_{x \to \pm\infty} \frac{\frac{2x^2-6x+1}{x^2}}{\frac{1+x^2}{x^2}} =$$

$$= \lim_{x \to \pm\infty} \frac{2 - \frac{6}{x} + \frac{1}{x^2}}{\frac{1}{x^2}+1} = \frac{2-0+1}{0+1} = 2$$

So $\frac{2x^2-6x+1}{1+x^2}$ has a horizontal asymptote at $y=2$

d) $$\lim_{x \to \pm\infty} \frac{x-1}{2x^2+1} = \lim_{x \to \infty} \frac{\frac{x-1}{x^2}}{\frac{2x^2+1}{x^2}} = \lim_{x \to \infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{2+\frac{1}{x^2}} = \frac{0-0}{2+0} = 0$$

so $\frac{x-1}{2x^2+1}$ has a horizontal asymptote of $y=0$.

8. Sketch the graph of $f(x) = \frac{x-1}{x+2}$

vertical asymptote: $x = -2$

horizontal asymptote $y = 1$

y-intercept: $(0, -\frac{1}{2})$

x-intercept: $(1, 0)$

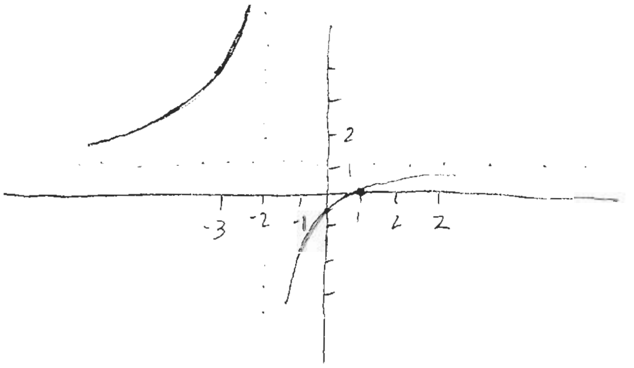
$$f'(x) = \frac{(x-1)'(x+2) - (x-1)(x+2)'}{(x+2)^2} = \frac{x+2-(x-1)}{(x+2)^2} = \frac{3}{(x+2)^2}$$

so $f'(x) > 0$ for $x \neq -2 \Rightarrow f(x)$ is increasing on $(-\infty, -2)$ and $(-2, \infty)$

$$f''(x) = 3(-2)(x+2)^{-3}(1) = \frac{-6}{(x+2)^2}$$

$f''(x) > 0$ on $(-\infty, -2)$ so $f(x)$ concave up on $(-\infty, -2)$

$f''(x) < 0$ on $(-2, \infty)$ so $f(x)$ concave down on $(-2, \infty)$



9. Find all points of inflection for $f(x) = x^3 - 12x$

$f'(x) = 3x^2 - 12$

$f''(x) = 6x$ so $f''(x)$ changes sign at $x = 0$

10. State the Mean Value Theorem

For $f(x)$ continuous on $[a,b]$ and differentiable on (a,b):
there exists c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$